De Moivre's theorem and nth roots

De Moivre's theorem is not only true for the integers but can be extended to fractions.

De Moivre's theorem for fractional powers

$$\{r(\cos\theta + i\sin\theta)\}^{p/q} = r^{p/q}\left\{\cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)\right\}$$

Example 1

Calculate $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3}$

By De Moivre's theorem for fractional powers

$$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3} = \left\{\cos\left(\frac{1}{3}\times\frac{\pi}{4}\right) + i\sin\left(\frac{1}{3}\times\frac{\pi}{4}\right)\right\} = \left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Example 2

Calculate $\left\{27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right\}^{1/3}$

By De Moivre's theorem

$$\left\{27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right\}^{1/3} = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} + i\,\frac{3}{2}$$

Example 3

Using De Moivre's theorem calculate $\left\{8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^{2/3}$

De Moivre's theorem gives $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\sqrt{3} + 2i$

The previous worked example showed that $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3} = \left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

That is, $\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ is a cube root of $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

This cube root is obtained by dividing the argument of the original number by 3

However, the cube roots of $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ are complex numbers z which satisfy $z^3 = 1$ and so by the Fundamental theorem of algebra, since this equation is of degree 3, there should be 3 roots. That is, in general, a complex number should have 3 cube roots.

Given a complex number these 3 cube roots can always be found

Strategy for finding the cube roots of a complex number

- Write the complex number in polar form z = r (cos θ+ i sin θ)
- Write z in two more equivalent alternative ways by adding 2π to the argument.
  z = r {cos (θ + 2π) + i sin (θ + 2π)}
  z = r {cos (θ + 4π) + i sin (θ + 4π)}
- Write down the cube roots of z by taking the cube root of r and dividing each of the arguments by 3

NB: the previous strategy gives the three cube roots as

$$r^{1/3}\left\{\cos\left(\frac{\theta}{3}\right)+i\sin\left(\frac{\theta}{3}\right)\right\}$$

$$r^{1/3}\left\{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)\right\}$$

$$r^{1/3}\left\{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)+i\sin\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)\right\}$$

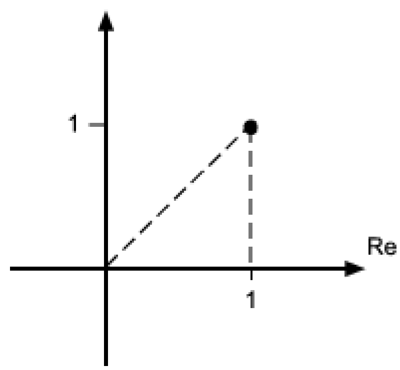
If z = r (cos θ+ i sin θ) is written in any further alternative ways such as
z = r {cos (θ + 6π) + i sin (θ + 6π)}, this gives a cube root of

$$r^{1/3}\left\{\cos\left(\frac{\theta}{3}+\frac{6\pi}{3}\right)+i\sin\left(\frac{\theta}{3}+\frac{6\pi}{3}\right)\right\}=r^{1/3}\left\{\cos\left(\frac{\theta}{3}\right)+i\sin\left(\frac{\theta}{3}\right)\right\}$$

which is the same as one of the previously mentioned roots.
It is impossible to find any more.

Example 4
Find the cube roots of 1 + i

First express 1 + i in polar form



$|1+i|=\sqrt{2}$ and arg (1 + i) = $\frac{\pi}{4}$

Hence1 + i can be expressed as $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

But 1 + i can also be expressed as

$$1+i=\sqrt{2}\left\{\cos\left(\frac{\pi}{4}+2\pi\right)+i\sin\left(\frac{\pi}{4}+2\pi\right)\right\}=\sqrt{2}\left\{\cos\left(\frac{9\pi}{4}\right)+i\sin\left(\frac{9\pi}{4}\right)\right\}$$

and

$$1+i=\sqrt{2}\left\{\cos\left(\frac{\pi}{4}+4\pi\right)+i\sin\left(\frac{\pi}{4}+4\pi\right)\right\}=\sqrt{2}\left\{\cos\left(\frac{17\pi}{4}\right)+i\sin\left(\frac{17\pi}{4}\right)\right\}$$

Hence, taking the cube root of the modulus and dividing the argument by 3, the cube roots of 1 + i are

$$z = (2^{1/2})^{1/3}\left(\cos\frac{\pi}{12} + i\ \sin\frac{\pi}{12}\right) = (2^{1/6})\left(\cos\frac{\pi}{12} + i\ \sin\frac{\pi}{12}\right)$$
$$z = (2^{1/6})\left\{\cos\left(\frac{3\pi}{4}\right)\ + i\ \sin\left(\frac{3\pi}{4}\right)\right\}$$
$$z = (2^{1/6})\left\{\cos\left(\frac{17\pi}{12}\right)\ + i\ \sin\left(\frac{17\pi}{12}\right)\right\}$$

In this way the nth roots of any complex number can be found.

Example 5

Find the cube roots of z = 64(cos 30° + i sin 30°)

Answer:
This is in polar form. Use 2π = 360° and 4π = 720°
z = 64(cos 30° + i sin 30°)
z can also be written as
z = 64{cos (30 + 360)° + i sin (30 + 360)°}
and
z = 64{cos (30 + 720)° + i sin (30 + 720)°}
Since $64^{1/3} = \sqrt[3]{64} = 4$, the cube roots of z are
4(cos 10° + i sin 10°), 4(cos 130° + i sin 130°), 4(cos 250° + i sin 250°)

Example 6

Find the fourth roots of 81i, that is of $81\left(\cos\frac{\pi}{2} + i\ \sin\frac{\pi}{2}\right)$

z = $81\left(\cos\frac{\pi}{2} + i\ \sin\frac{\pi}{2}\right)$
z = $81\left(\cos\frac{5\pi}{2} + i\ \sin\frac{5\pi}{2}\right)$
z = $81\left(\cos\frac{9\pi}{2} + i\ \sin\frac{9\pi}{2}\right)$
z = $81\left(\cos\frac{13\pi}{2} + i\ \sin\frac{13\pi}{2}\right)$
$r^{1/4}$= 3

The fourth roots of $81\left(\cos\frac{\pi}{2} + i\ \sin\frac{\pi}{2}\right)$are

$$3\left(\cos\ \frac{\pi}{8} + i\ \sin\frac{\pi}{8}\right), 3\left(\cos\ \frac{5\pi}{8} + i\ \sin\frac{5\pi}{8}\right),$$
$$3\left(\cos\ \frac{9\pi}{8} + i\ \sin\frac{9\pi}{8}\right)\ \text{and}\ 3\left(\cos\ \frac{13\pi}{8} + i\ \sin\frac{13\pi}{8}\right)$$

Example 7
Find the sixth roots of $\sqrt{3} + i$

The modulus of $\sqrt{3} + i$ is 2 and the argument is $\frac{\pi}{6}$

The sixth roots are $\sqrt[6]{2}\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)$, $\sqrt[6]{2}\left\{\cos\left(\frac{13\pi}{36}\right) + i\sin\left(\frac{13\pi}{36}\right)\right\}$,
$\sqrt[6]{2}\left\{\cos\left(\frac{25\pi}{36}\right) + i\sin\left(\frac{25\pi}{36}\right)\right\}$, $\sqrt[6]{2}\left\{\cos\left(\frac{37\pi}{36}\right) + i\sin\left(\frac{37\pi}{36}\right)\right\}$,
$\sqrt[6]{2}\left\{\cos\left(\frac{49\pi}{36}\right) + i\sin\left(\frac{49\pi}{36}\right)\right\}$ and $\sqrt[6]{2}\left\{\cos\left(\frac{61\pi}{36}\right) + i\sin\left(\frac{61\pi}{36}\right)\right\}$

It is easy and important to find the nth roots of 1
i.e. complex numbers such that $z^n = 1$
Such numbers are often referred to as the nth roots of unity.

> **Roots of unity**
>
> The nth roots of unity are those numbers that satisfy the equation $z^n = 1$

Since $1 = \cos 2\pi + i \sin 2\pi$, it follows that $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is an nth root of unity.
But 1 can be written using different arguments as follows:

$$
\begin{aligned}
1 &= \cos 2\pi + i\sin 2\pi \\
&= \cos 4\pi + i\sin 4\pi \\
&= \cos 6\pi + i\sin 6\pi \\
&= \ldots\ldots\ldots\ldots\ldots\ldots \\
&= \cos 2n\pi + i\sin 2n\pi
\end{aligned}
$$

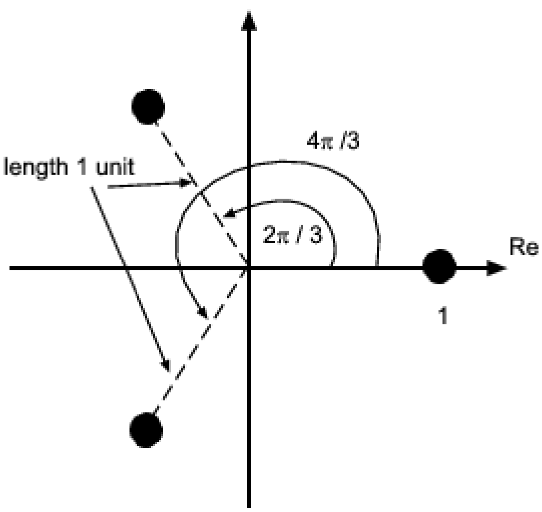
Hence dividing the argument in each case by n gives the following nth roots of unity.

$$z = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$
$$z = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}$$
$$z = \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n} \text{ and so on.}$$

Note that arguments increase by $\frac{2\pi}{n}$ each time. The roots of unity are regularly spaced in an Argand diagram.

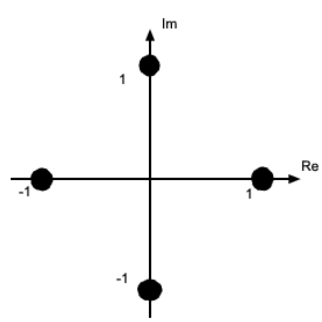
Example 8
Find the cube roots of unity and plot them on an Argand diagram.

Answer:
Since 1 can be written in polar form as
1 = cos 2π + i sin 2π
1 = cos 4π + i sin 4π
1 = cos 6π + i sin 6π
the cube roots of unity are
z = $\cos\frac{2\pi}{3} + \text{i}\ \sin\frac{2\pi}{3} = \frac{1}{2}(-1 + \text{i}\sqrt{3})$
z = $\cos\frac{4\pi}{3} + \text{i}\ \sin\frac{4\pi}{3} = \frac{1}{2}(-1 - \text{i}\sqrt{3})$
z = $\cos\frac{6\pi}{3} + \text{i}\ \sin\frac{6\pi}{3} = 1$
On the Argand diagram



Example 9
Find the fourth roots of unity and plot them on an Argand diagram.

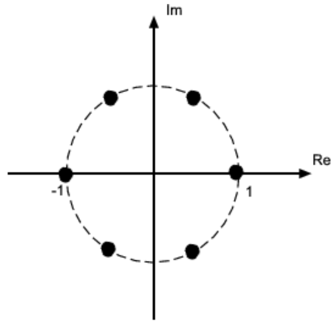
The solutions are
z = cos 0 + i sin 0 , $\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}$, cos π+ i sin $\pi$and $\cos\frac{3\pi}{2} + \text{i}\sin\frac{3\pi}{2}$
i.e. z = 1, -1, i and -i
On an Argand diagram this gives



Example 10
Find the solutions of the equation $z^6 - 1 = 0$. Plot the answers on an Argand diagram.

The solutions are
$\cos 0 + i \sin 0$, $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$, $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$, $\cos \pi + i \sin \pi$,
$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$ and $\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$

The Argand diagram gives



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