Lines, Intersections, Regions

Aim
To investigate the relationships that exists when lines are drawn on a plane.

Strategies
To develop some conjectures, rules and patterns by investigating the relationships that form between the number of lines, intersections points, and bounded and unbounded regions. I plan to use rules to further define the relationship.

The following relationships will be investigated:

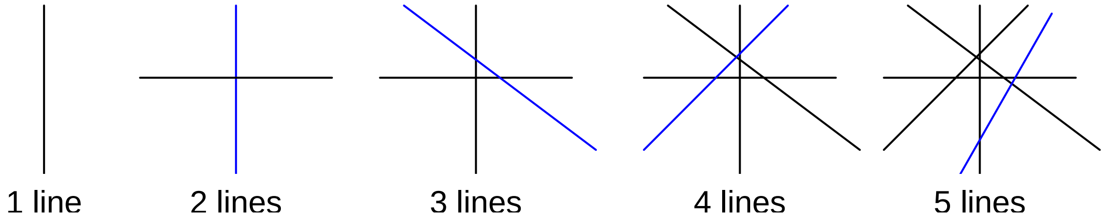
# Lines-	# unbounded regions
# Lines-	min # bounded shapes
# Lines-	max # bounded shapes
# Lines	# possibilities of intersections
# Lines-	max. # Intersections
# Intersects-	# bounded regions
# Lines-	min # bounded triangles
# Lines-	max # bounded triangles
# Lines-	# bounded regions (not perpendicular lines)
# Lines-	# bounded quads

General Rules:

1. When considering minimum regions formed, lines cannot be drawn parallel. After the first 2 lines have been drawn, it is possible for region(s) to form using the 3rd line. When investigating minimum regions, at LEAST 1 region must be created with every line.
2. Two consecutive lines cannot be drawn parallel
3. If a line crosses through an intersection already formed by other lines, it does not count as an extra point of intersection
4. *n:* number of lines
   y: Number of bounded or unbounded regions
   x: number of intersects

When *x* number of lines are drawn, how many unbounded regions are formed?

***RULES**: 1. The first 2 lines drawn must be perpendicular*
2. The line drawn must go from one side all the way across to the other (it cannot only cross halfway)



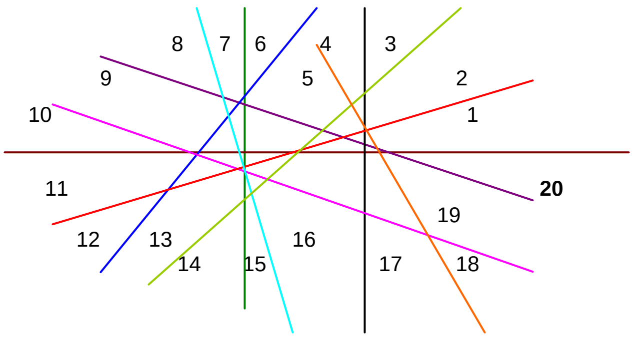
# Lines	# Unbounded Regions	# Bounded Regions
1	2	0
2	4	0
3	6	1
4	8	2
5	10	3

<u>Conjecture A</u>
The data above suggests that there is a relationship between the number of lines drawn and the number of unbounded regions formed. When 1 line is drawn, there are 2 unbounded regions. When 2 lines are drawn, 4 unbounded regions form. The number of unbounded regions formed is always double the number of lines.

∴ **y=2n** (where *y* is the number of unbounded regions)

<u>Testing Conjecture A</u>
How many unbounded regions will form when 10 lines are drawn?

<u>Prediction</u>
y=2n
y=2 x 10
y=20



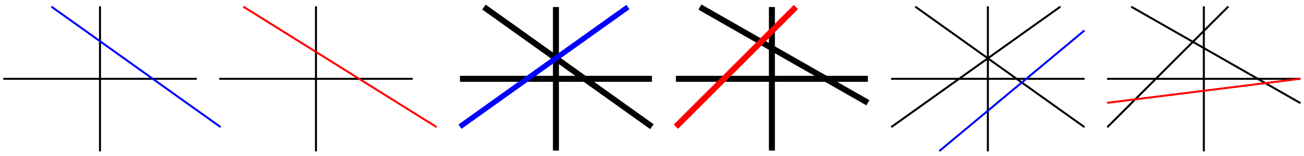
<u>Justifying Conjecture A</u>
The number of unbounded regions in double the number of lines because every time another line is drawn, it is splitting the diagram creating 2 more regions. As another 2 regions are created with every line added, 2 more regions are added to the whole diagram.

When investigating Conjecture A, the data produced showed that another relationship exists between the number of lines and the number of bounded regions.

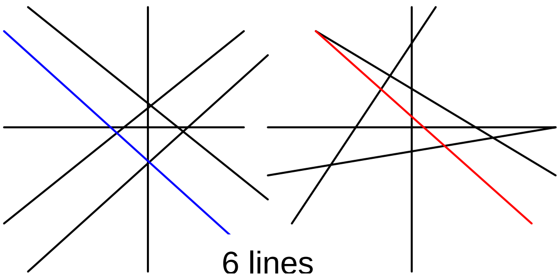
# Lines	# Bounded Regions MIN	# Bounded Regions MAX
1	0	0
2	0	0
3	1	1
4	2	3
5	3	6
6	4	10
7	5	15
8	6	21

<u>*Conjecture B*</u>

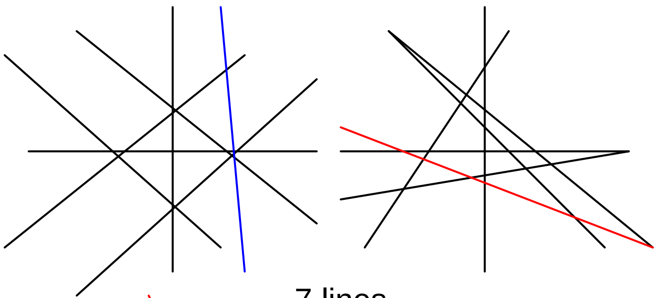
What is the relationship between the number lines and the minimum and maximum number of bounded shapes formed?



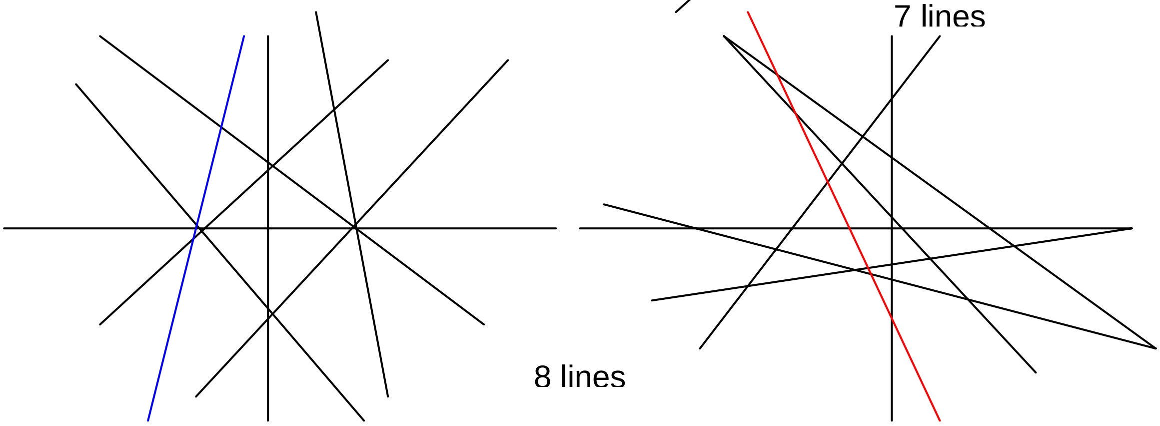
3 lines 4 lines 5 lines



6 lines



7 lines



8 lines

From the table of results, it can be observed that the number of lines subtract 2 equals to the minimum number of bounded regions.

∴ The relationship between the number of lines and minimum regions is:

$y = n - 2$ (where y is the *minimum* number of regions formed)

The maximum number of bounded regions can be observed to be increasing the first time by 1 region, then 2, then 3, etc. Because of this pattern, a relationship must exist between the number of lines and the maximum regions. As it does not have a common difference (it does not increase by the same amount each time), a table can be constructed to see if there is a second common difference (in which case, the relationship will be quadratic).

n	**0**	1	2	3	4	5	6	7	8
y	**1**	0	0	1	3	6	10	15	21

-1 0 1 2 3 4 5 6

1 1 1 1 1 1 1

$y = a x^2 + b x + c$

a = ½ the 2nd common difference
(1 x ½)= **½**

b= (a+b)= the common diff. between 0 and 1
(½ + b)= -1 ∴ b= **- $^3/_2$**

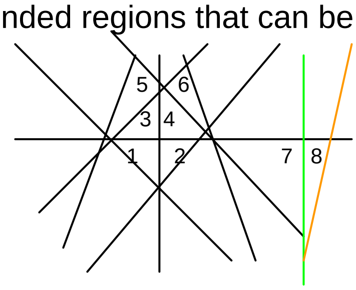
c= when n is 0, find y
n=0 ; y=**1**

Therefore the relationship between the number of lines and maximum regions is:

$$y = {}^1/_2\, n^2 - {}^3/_2\, n + 1$$

Testing Conjecture B

What is the minimum number of bounded regions that can be form using 10 lines?



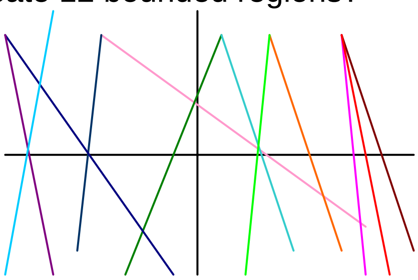
<u>Prediction</u>

y=n – 2

=10-2

=**8 regions**

How many lines will be needed to create 12 bounded regions?



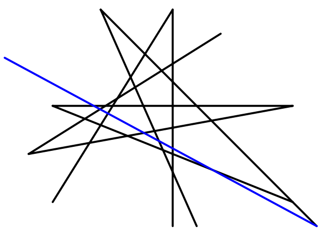
<u>Prediction</u>

y=n - 2

y + 2 = n

12 + 2= **14 lines**

What is the maximum number of bounded regions that can be formed using 9 lines?



Prediction

$Y = ½ x^2 - {}^3/_2\, x + 1$

$= ½ (9^2) - {}^3/_2 (9) + 1$

= **28 regions**

Justifying Conjecture B

The minimum regions equals the number of lines subtract 2 because the first two lines drawn create no regions, but from then on, only 1 region is formed. (eg: 3 lines, 1 region). As each lines increases by 1, the region also increases by 1. So, because of the 2 lines at the start creating no regions, the pattern continues.

Conjecture C

From the table created previously (where relationships between the number of lines, and the minimum and maximum number of regions were found), there appears to be another **relationship between the maximum and minimum number of regions.** The maximum number of regions increases by 0, then 1, then 2, then 3, etc. A second common difference of 1 exists, presenting a quadratic relationship.

# Lines	**# Bounded Regions MIN**	**# Bounded Regions MAX**
1	**0**	**0**
2	**0**	**0**
3	**1**	**1**
4	**2**	**3**
5	**3**	**6**
6	**4**	**10**
7	**5**	**15**
8	**6**	**21**

n	0	1	2	3	4	5	6
y	0	1	3	6	10	15	21
	1	*2*	*3*	*4*	*5*	*6*	
	1	***1***	***1***	***1***	***1***		

$$y = a x^2 + b x + c$$

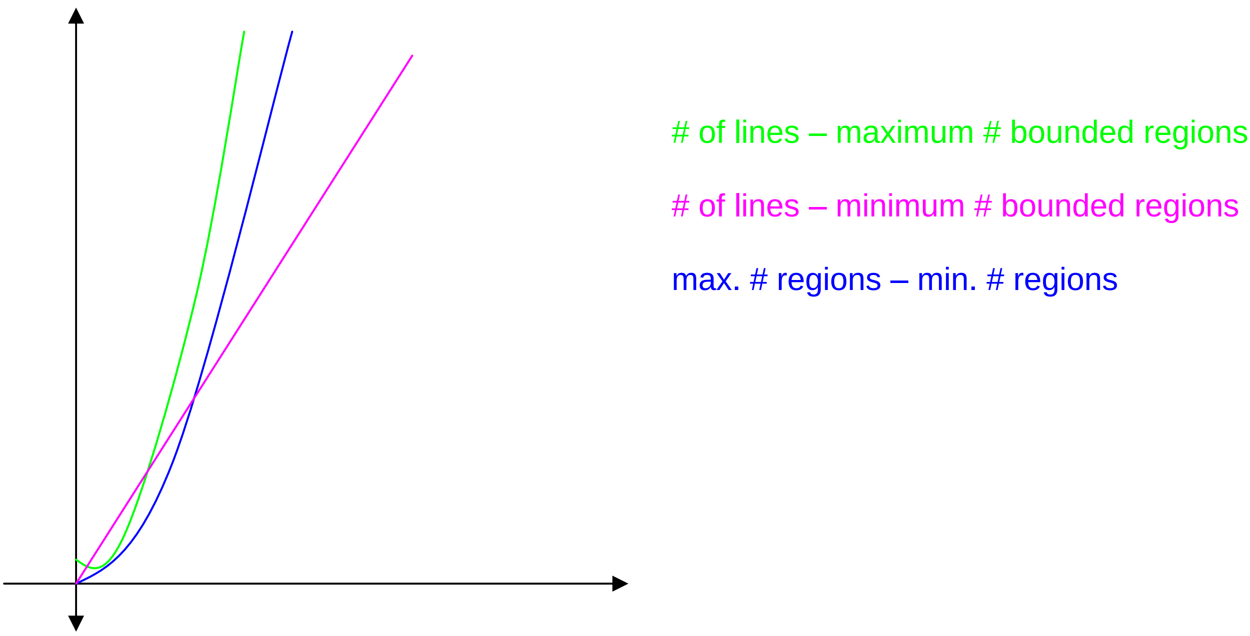
a = ½ the 2nd common difference
(1 x ½)= ½

b= (a+b)= the common diff. between 0 and 1
(½ + b)= 1 ∴ b= ½

c= when n is 0, find y
n=0 ; y=**0**

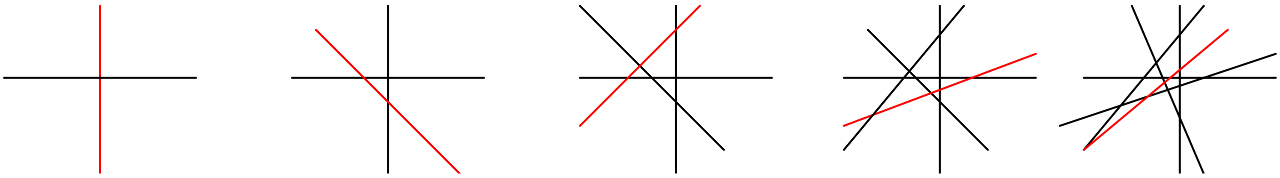
∴The quadratic relationship between the minimum and maximum number of regions is:

$$y = \frac{1}{2} n^2 + \frac{1}{2} n$$



Conjecture D

What is the number of possible intersections that can be created when lines are drawn across a set of perpendicular axis?



2 lines 3 lines 4 lines 5 lines 6 lines

# of lines	# of possible intersections	Max. # Intersections
0	0	0
1	0	0
2	1	1
3	1 , 2 , 3	3
4	1 , 2 , 3 , 4 , 5 , 6	6
5	1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10	10
6	1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10 , 11 , 12 , 13 , 14 , 15 ,	15

Again the number of possible intersections (maximum number of intersections) is increasing by 1, then 2, then 3, etc. There is a second common difference of 1, from which we can tell from previous conjectures, will result in a quadratic relationship.

$y = a x^2 + b x + c$

n	**0**	1	2	3	4	5	6
y	**0**	0	1	3	6	10	15
	0	*1*	*2*	*3*	*4*	*5*	
	1	***1***	***1***	***1***	***1***		

a = ½ the 2nd common difference
(1 x ½)= **½**

b= (a+b)= the common diff. between 0 and 1
(½ + b)= 0 ∴ b= **- ½**

c= when n is 0, find y
n=0 ; y=**0**

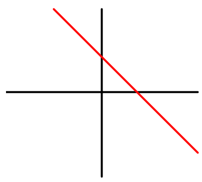
∴The relationship between number of possible intersection points (y) and the number of lines (n) is:

$$y = \frac{1}{2} n^2 - \frac{1}{2} n$$

Justifying Conjecture D

The number of maximum intersection points always increases by the previous number plus one. This is because, for the maximum number of points to form, the new line must cross through **all** the previous lines drawn, creating the same number of intersection points as the lines it has crossed through. That number of points is then added to the points created before.

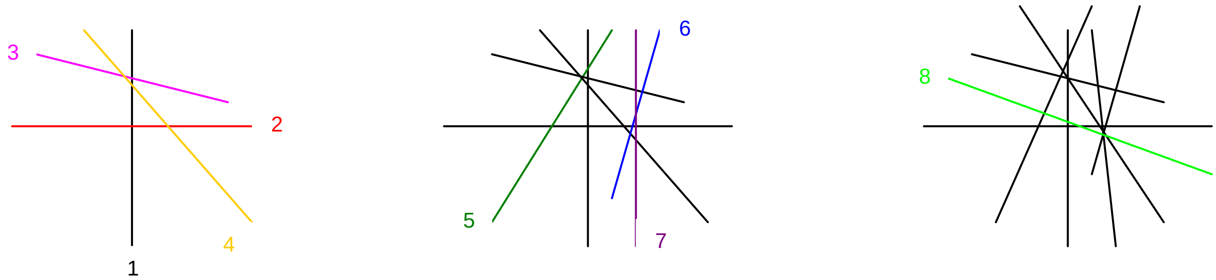
Eg: A 3rd line is added to intersect with the 2 lines already drawn. It crosses. Those 2 lines (creating 2 intersections) *plus* the one intersection created before, which therefore equals 3.



Conjecture E

Relationships have been found to exist between lines and bounded regions, lines and intersection points. **Is there a pattern between the number of bounded regions and intersection points?**

***RULES:**1. In this particular conjecture, bounded regions must form triangles.*



# of lines	**# of intersections**	**# of bounded regions △**
2	**1**	**0**
3	**2**	**0**
4	**3**	**1**
5	**4**	**2**
6	**5**	**3**
7	**6**	**4**
8	**7**	**5**

Excluding the 1st intersection, the number of bounded triangles is 2 less than the number of intersection points. (eg: 6 intersection points created 4 bounded regions). The difference between the bounded regions is common (increasing by 1 with each new intersection), therefore the relationship between the number of intersections and bounded triangles is linear:

$$\mathbf{y = x - 2}$$

Testing Conjecture E

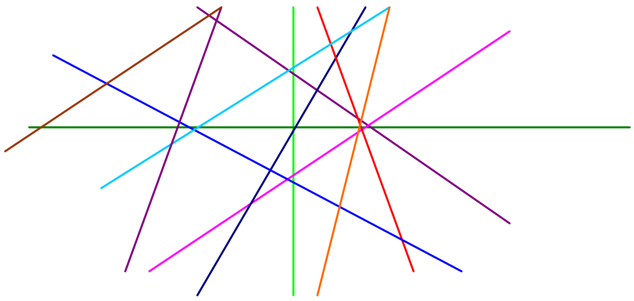
How many bounded regions(△) will form when there are 18 intersections?

Prediction

y = *x* – 2

y = 18 – 2

y = 16

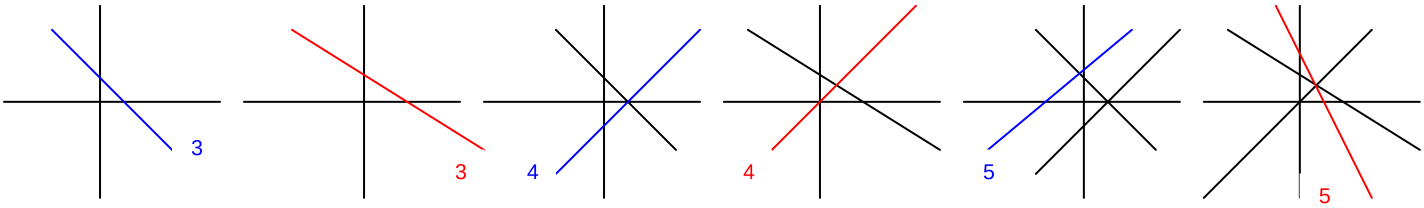


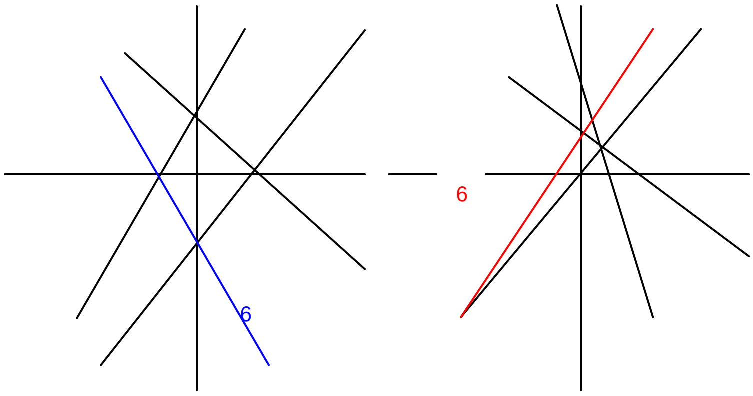
Justifying Conjecture E

The reason the number of bounded triangles equals the number of intersection points subtract 2 is similar to the relationship found in Conjecture B (between lines and minimum regions). The first 2 points create no regions, but after that, when every 1 line intersects the others, 1 regions is formed. A triangle need three side to form, so after the first 2 lines intersect, a 3rd line can be drawn creating another point, creating another region.

Conjecture F

What is the relationship between the number of lines and the minimum and maximum number of TRIANGLES created?





# of lines	Min. # of △	Max. # of △
1	0	0
2	0	0
3	1	1
4	2	2
5	3	4
6	4	7

The relationship between lines and the minimum number of TRIANGLES formed is: **$y = n - 2$** The same rules and justifications relate to this conjecture as to conjecture B (where the relationship between the number of lines and the minimum bounded regions of any shape was investigated.

The maximum number of TRIANGLES formed (excluding the 1st and 2nd line drawn) increases by the previous difference plus one; it increases by 1, then 2, then 3, etc. Again a second common difference can be observed, resulting in a quadratic relationship.

n	0	1	2	**3**	**4**	**5**	**6**
y	0	0	0	**1**	**2**	**4**	**7**
	-2	*-1*	*0*	*1*	*2*	*3*	
	1	*1*	*1*	***1***	***1***		

$y = a x^2 + b x + c$

a = ½ the 2nd common difference

$(1 \times \frac{1}{2}) = \frac{1}{2}$

b= (a+b)= the common diff. between 0 and 1

$(\frac{1}{2} + b) = -2 \quad \therefore b = -\frac{5}{2}$

c= when n is 0, find y

n=0 ; y=?

The 'c' value (y-intercept) cannot be found using the table because the equation excludes the 1st and 2nd value for n (which are the first 2 lines drawn)

Hence, to find the 'c' value, the equation must be re-arranged to solve for 'c'. To do this, a number must be substituted in for *y* (which is the number of bounded regions), and also for *n* (number of lines).

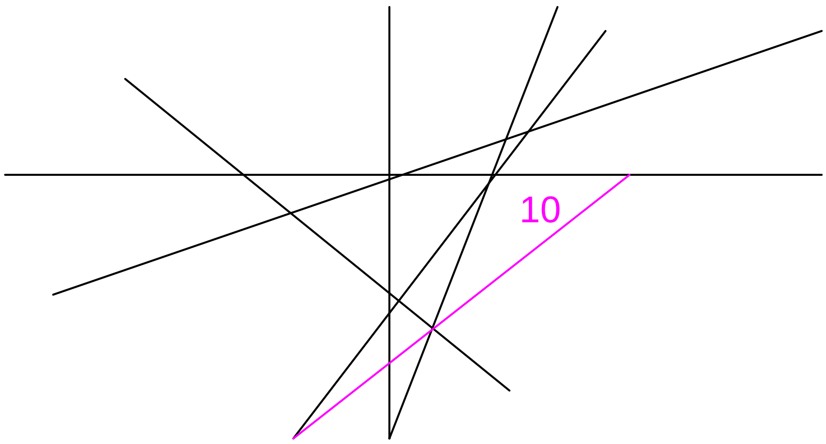
$y = a x^2 + b x + c$
$y = ½ n^2 - {}^5/_2 n + c$ (Record what values of the equation are known)
$1 = ½ (3^2) - {}^5/_2(3) + c$ (Substitute in the value for *y*, and for n)
$c = 1 - ½ (3^2) + {}^5/_2 (3)$ (Re-arrange equation to solve for *c*)
c = 4

∴The rule determining the number of bounded TRIANGLES (y) formed when (n) number of lines are drawn is:

$$y = {}^1/_2 n^2 - {}^5/_2 n + 4$$

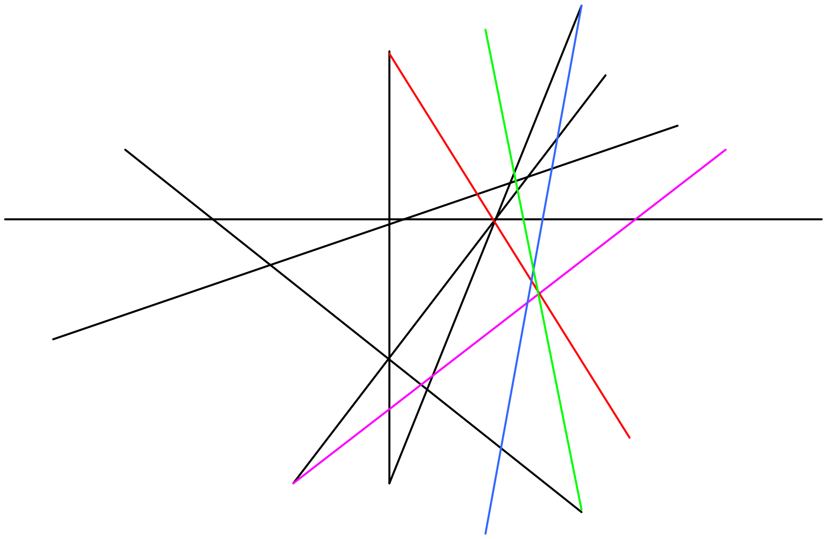
Justifying Conjecture F
The **minimum number of triangles** formed is 2 less than the number of lines used to create them because the first 2 lines create no triangles, but from then on, 1 triangle is created. So as the number of lines increase by 1, so does the number of bounded triangles.

Testing Conjecture F
Prediction
$y = {}^1/_2 n^2 - {}^5/_2 n + 4$
y = 11



Conjecture False!
The rule relating the number of Triangles formed when lines are drawn does not work any further than 6 lines. When the 7th line is drawn, only 10 regions are formed, not 11, as the conjecture states. The following table states the maximum number of triangles formed using more than 6 lines:

# of Lines	Max. # of Triangles
7	10
8	13
9	17
10	22



n	0	1	2	3	4	5	6	**7**	**8**	**9**	**10**
y	17	13	10	8	7	7	8	**10**	**13**	**17**	**22**

-4 - 3 - 2 - 1 0 1 2 3 4 5

1 1 1 1 1 1 1 1 1

If the table is constructed in a way in which it includes the values of lines from 7 –10, then it can help to produce the quadratic equation/rule.

$y = a x^2 + b x + c$

a = ½ the 2nd common difference

(1 x ½)= ½

b= (a+b)= the common diff. between 0 and 1

(½ + b)= -4 ∴ b= -4.5

c= when n is 0, find y

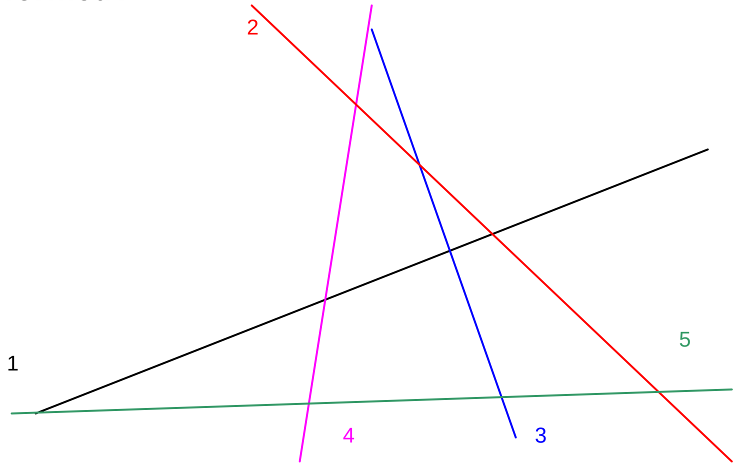
n=0 ; y=17

∴The rule connecting the maximum number of triangles produced when more than 6 lines a drawn is:

$y = 0.5 n^2 - 4.5 n + 17$

Conjecture G

What is the relationship between lines (not perpendicular) and the maximum number of bounded regions formed?



The number of bounded regions has a second common difference. That suggests that the relationship between the number of lines drawn and the maximum number of bounded regions created is a quadratic one.

# of Lines	# of Bounded Regions
1	0
2	0
3	1
4	3
5	6

n	1	2	3	4	5
y	0	0	1	3	7

0 1 2 4

-1 1 1

$y = a x^2 + b x + c$

a = ½ the 2nd common difference

(1 x ½)= ½

b= (a+b)= the common diff. between 0 and 1

(½ + b)= -1 ∴ b= $-^3/_2$

c= when n is 0, find y

n=0 ; y=1

∴ The rule that connects the number of lines (that are not perpendicular) to the number of bounded shapes the create is:

$$y = {}^1/_2 n^2 - {}^3/_2 n + 1$$

Testing Conjecture G

Using the rule, find out how many bounded regions will form when **6 lines** are drawn.

Prediction

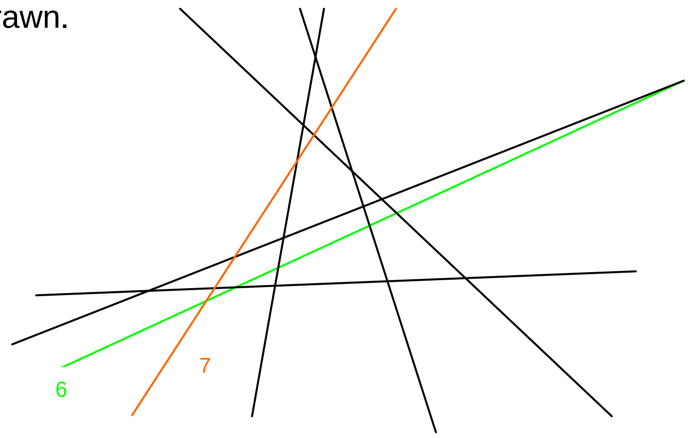
$y = {}^1/_2 n^2 - {}^3/_2 n + 1$

y = 10 regions

When **7 lines** are drawn?

Prediction

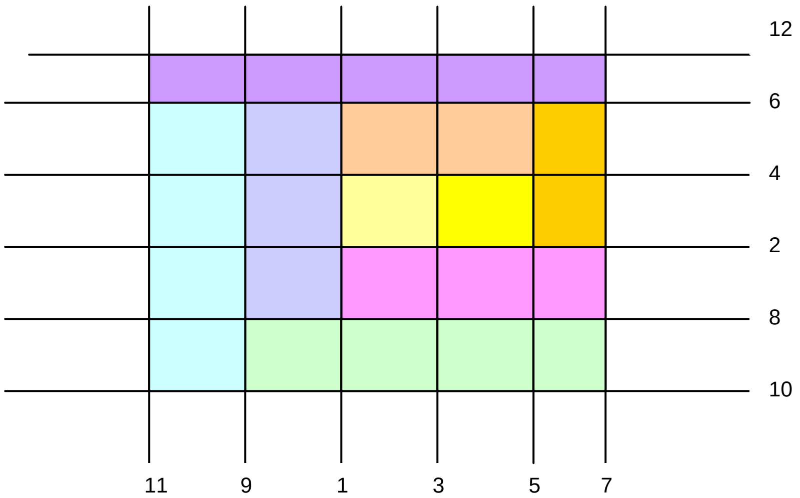
y = 15 regions



Conjecture H

How many quadrilateral regions form when perpendicular lines form a grid?

***RULES:** 1. Two consecutive lines cannot go in the same directions (One line must go horizontal, then vertical, horizontal, then vertical).
2.Lines must intersect at 90^0, forming quadrilaterals.



# of Lines (n)	# of Quads Formed (y)
1	0
2	0
3	0
4	1
5	2
6	4
7	6
8	9
9	12
10	16
11	20
12	25

n	1	2	3	4	5	6	7	8	9	10	11	12
y	0	0	0	1	2	4	6	9	12	16	20	25

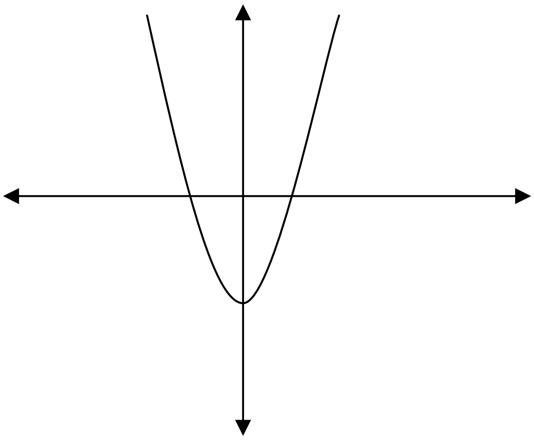
0 1 1 2 2 3 3 4 4 5

1 0 1 0 1 0 1 0 1

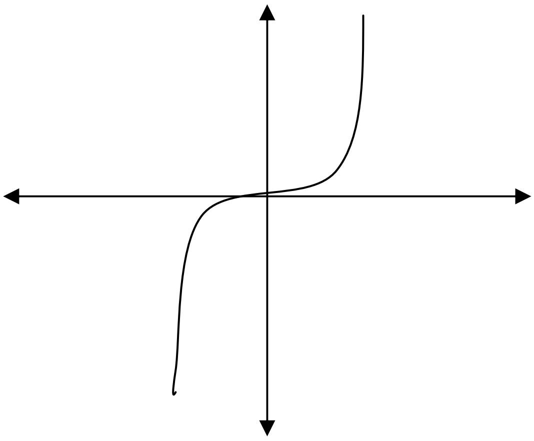
1 1 1 1 1 1 1 1

When organised into a table of values, a third common difference of 1 appears. This difference suggests that a **cubic relationship** is present.

However, when the values for n and y are graphed, the graph produced resembles that of a quadratic equation (parabola).



Quadratic (parabola)



Cubic Graph

To find the equation, we can substitute the values from the table, and solve the equation by trial and error.

(Take the coordinates 5 (n) and 2 (y))

$y = a x^2 + bx + c$
$2 = 5^2 + 5 + c$
$2 = 30 + c$

Try just substituting in values for x and y, without considering any values for a or b

$y = a x^2 + bx + c$
$2 = 5^2 - 5 + c$
$2 = 20 + c$

Try subtracting bx, instead of adding it.

$y = a x^2 + bx + c$
$2 = (½)5^2 – 5 + c$
$2 = 7.5 + c$
$-5.5 = c$

Then try adding numbers in front of *a*. This gives an equation of <u>$y = (½)x^2 + bx – 5.5$</u>

$y = (½)x^2 + x – 5.5$
$y = (½)3^2 – 3 – 5.5$
$y = 2$

To see if this equation fits, substitute other values from the table (eg: 3 and 1)

From the previous trials and errors, we can see that the equation is almost there (When substituting 3, the equation gave an answer of 2 for *y*, instead of 1).

$y = a x^2 - bx + c$
$2 = (¼)5^2 – 5 + c$
$2 = 6.25 – 5 + c$
$2 = 1.25 + c$
<u>$c = ¾$</u>

Try making the value of *a* even smaller.

$\mathbf{y = (1/4)\,x^2 - x + 3/4}$
$y = (1/4)\,5^2 - 5 + 3/4$
$y = 6.25 - 5 + 3/4$
$y = 1.25 + 3/4$
$\mathbf{y = 2}$

Now, substitute 5 into this equation, to see if *y* will equal 2

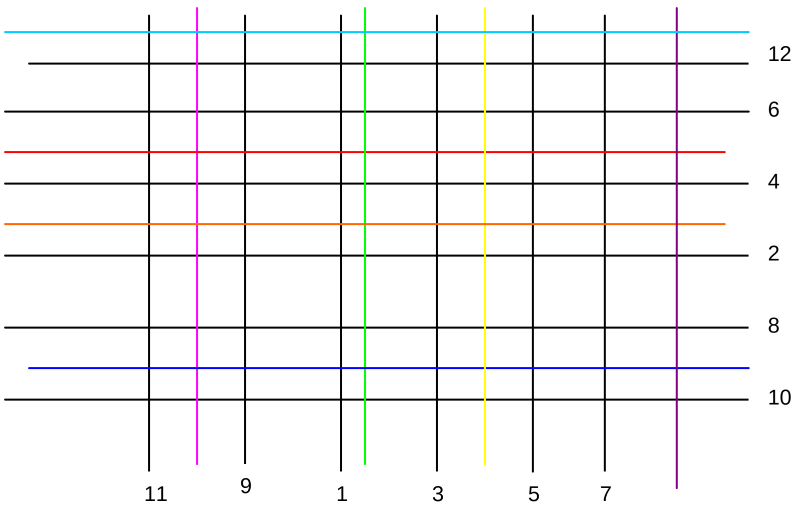
∴. The equation relating the number of lines and the number of quadrilateral regions are formed is:

$$y = \frac{1}{4} n^2 - n + \frac{3}{4}$$

NOTE: When graphed on the calculator, a more exact equation can be found. When using this equation, values for *y* must be rounded to the nearest whole number to show how many regions are formed.

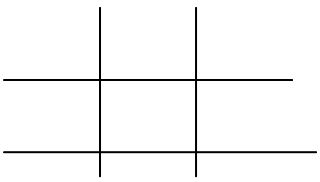
$$y = 0.25n^2 - 0.99n + 0.81$$

Testing Conjecture H
How many quadrilaterals will form when 20 lines are drawn?

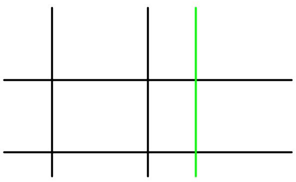


Prediction
$\mathbf{y = 0.25n^2 - 0.99n + 0.81}$
y = 0.25 (202)-0.99(20).0.81
y =80.81
y = 81 regions

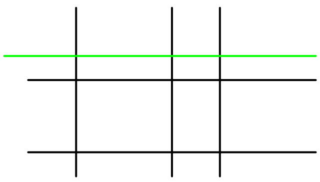
Justifying Conjecture H
When 4 lines are drawn, 1 quadrilateral forms, when there are 5 lines, 2 quadrilaterals. When more lines are added (one horizontally, then vertically), the number of new regions created depends on how many regions were formed before:



With 4 lines, 1 region is formed



When a 5th line is added vertically, the number of new regions is the same as however many are in the **column**



The 6th line created as many new regions as there are in the **row** *(2 more regions)*

Summary: "Lines, Intersections, Regions"
Investigating the relationships between lines, points of intersections, bounded and unbounded regions produced many patterns. These patterns and conjectures can be expressed by rules that can then be applies to determine values. The following are the rules that I observed during my investigation:

RELATIONSHIP		**RULE**
# Lines	# Unbounded Regions	y=2n
# Lines	Min. # Bounded Regions	y= n – 2
# Lines	Max. # Bounded Regions	$y = \frac{1}{2} n^2 - \frac{3}{2} n + 1$
# Lines	Max. # Intersections	$y = \frac{1}{2} n^2 + \frac{1}{2} n$
# Lines	# Possible Intersections	y = x – 2
# Lines	Min. # Bounded Triangles	y= n – 2
# Lines	Max. # Bounded Triangles	$y = \frac{1}{2} n^2 - \frac{5}{2} n + 4$
	OR	$y = 0.5 n^2 - 4.5 n + 17$
# Lines (not perpend.)	# Bounded Regions	$y = \frac{1}{2} n^2 - \frac{3}{2} n + 1$
# Lines	# Bounded Quads	$y = 0.25n^2 - 0.99n + 0.81$
# Intersections	# Bounded Regions	$y = \frac{1}{2} n^2 - \frac{1}{2} n$

WHAT I HAVE LEARNT
From this investigation, I found that something as basic as lines intersecting and creating shapes could produce many rules, patterns and conjectures. I found both linear and quadratic relationships. This investigation allowed me to explore some other functions as well, such as the Cubic Relationship (ax^3+bx^2+cx+d) by looking at the *third* common difference when constructing quadrilaterals with perpendicular lines.

STRATEGIES EXPLORED
I used a few problem-solving strategies throughout this investigation, such as producing linear equations from a table of values, quadratic functions from a table and also trial and error. During this investigation, I have explored some of these rules and graphs on my calculator to produce more accurate examples (such as the rule for the relationship between the number perpendicular lines and the number of quadrilaterals formed).

PROBLEMS
While conducting this investigation, I have come across problems. One problem that I have not come to a conclusion with (but one which I will work on and explore further) is the third common difference pattern, which produced a quadratic relationship. I have also come across one conjecture that proved to be false (for the relationship between lines and the maximum triangles formed. However, I attempted to solve this by producing another table of values, which then showed another quadratic relationship.